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A PRECONDITIONED CONJUGATE GRADIENT UZAWA-TYPE METHOD FOR THE SOLUTION OF THE STOKES PROBLEM BY MIXED Q1–P0 STABILIZED FINITE ELEMENTS

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SUMMARY

We study the behaviour of a conjugate gradient Uzawa-type method for a stabilized finite element approximation of the Stokes problem. Many variants of the Uzawa algorithm have been described for different finite elements satisfying the well-known Inf–Sup condition of Babuška and Brezzi, but it is surprising that developments for unstable 'low-order' discretizations with stabilization procedures are still missing. Our paper is presented in this context for the popular (so-called) Q1–P0 element.

First we show that a simple stabilization technique for this element permits us to retain the property of a convergence factor bounded independently of the discretization mesh size. The second contribution of this work deals with the construction of a less costly preconditioner taking full advantages of the block diagonal structure of the stabilization matrix. Its efficiency is supported by 2D and 3D numerical results.

KEY WORDS Stokes equations Mixed finite elements Stabilization Uzawa-type algorithm Preconditioning

1. INTRODUCTION

Many numerical methods have been presented for the solution of the mixed finite element approximation of the Stokes problem which leads to a linear system of the following form:

$$\begin{pmatrix} A & B^{\mathrm{T}} \\ B & 0 \end{pmatrix} \begin{Bmatrix} u \\ p \end{Bmatrix} = \begin{Bmatrix} f \\ 0 \end{Bmatrix}. \tag{1}$$

Among these methods, one can choose a variant of the well-known Uzawa algorithm[1] by applying a conjugate gradient method to the solution of the dual problem associated with (1):

$$(B A^{-1} B^{\mathrm{T}}) \{p\} = B A^{-1} f. \tag{2}$$

This method has been described and improved by numerous authors[2–4] for stable discretizations by mixed finite elements satisfying the Inf–Sup condition of Babuška[5] and Brezzi.[6] In this context one can assume existence and uniqueness of a solution to (1) and can retain the main property that the condition number of $B A^{-1} B^{\mathrm{T}}$ is bounded independently of the mesh size.

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Unfortunately, for less costly, 'low-order' finite elements such as the popular Q1–P0, this Inf–Sup condition is not satisfied and thus, among other troubles arising from instability, iterative methods applied to (1) or (2) are totally inefficient (see References 3 and 7 and Figure 3).

In order to overcome the difficulties related to unstable discretization, numerous works have dealt with the proposal of 'low-order' compatible finite elements (mini-elements[8,9]) or with the stabilization of those not satisfying the Inf–Sup condition, i.e. P1–P1, Q1–Q1 and Q1–P0.[7,10–16] In most of these stabilization or condensation techniques one replaces the discrete incompressibility constraint $Bu=0$ by $Bu-Cp=0$, where C is a regularization matrix,[7,10–12,14–16] and the initial mixed problem (1) then becomes

$$\begin{pmatrix} A & B^{\mathrm{T}} \\ B & -C \end{pmatrix} \begin{Bmatrix} u \\ p \end{Bmatrix} = \begin{Bmatrix} f \\ 0 \end{Bmatrix}. \tag{3}$$

With benefits from the stabilization, multigrid methods have been successfully used for the solution of (3),[7,8] but it is surprising that no developments of the Uzawa algorithm have been presented for the solution of the *stabilized dual problem* associated with (3):

$$(B A^{-1} B^{\mathrm{T}} + C)\,\{p\} = B A^{-1} f. \tag{4}$$

In the present paper we want to develop this approach and to prove that the breakdown of Q1–P0 in the Uzawa algorithm[3] can be overcome by taking full advantage of the operator of stabilization.[16] This leads to an efficient algorithm for the lowest compatible rectangular finite element whose simplicity and 'low cost', in terms of either storage requirements or CPU time, are of main interest, especially for 3D computations.

The remainder of this paper is organized as follows. Sections 2 and 3 deal with the Stokes equations, the Q1–P0 discretization and the stabilization procedure. A criterion for the choice of the operator C is given, as proposed in Reference 16, and the 'local jump formulation' of Kechkar and Silvester is adopted. In Section 4 we concentrate our attention on Uzawa-type algorithms for the solution of (2) or (4). A comparison between these two variants clearly shows the benefits of the stabilization, which permit one to retain the main property that the operator $B A^{-1} B^T + C$ is positive definite and has a condition number bounded independently of the mesh size, which of course may not be assumed in the non-stabilized case (2). For the solution of (4) we apply a standard conjugate gradient method improved by using the preconditioner[17]

$$\bar{L} = \mathrm{diag}(B \bar{A}^{-1} B^{\mathrm{T}}) + C, \quad \text{where } \bar{A} = \mathrm{diag}(A). \tag{5}$$

This less costly preconditioner does not require the exact solution of a 'pressure Laplacian-type equation'[2] for the pressure and makes the most use of the block diagonal structure of the matrix C. Its efficiency is supported by two- and three-dimensional experiments for the test problem of the lid-driven cavity.

2. THE FINITE ELEMENT DISCRETIZATION

Let us recall the velocity–pressure formulation of the steady state Stokes equations given in its simplest form:

$$-\Delta u + \mathrm{grad}(p) = f \quad \text{in } \Omega, \tag{6a}$$

$$\mathrm{div}\, u = 0 \quad \text{in } \Omega, \tag{6b}$$

$$u = 0 \quad \text{on } \Gamma, \tag{6c}$$

$$\int_\Omega p \, \mathrm{d}\Omega = 0, \tag{6d}$$

where Ω is an open 'regular' bounded domain of $\mathbb{R}^d$ ($d=2$ or 3) and Γ its boundary and where u and p represent respectively the velocity field and the pressure field of an incompressible viscous (with viscosity equal to unity) fluid subject to exterior forces f.

If non-homogeneous boundary conditions are used ($u=g$ on Γ), they must satisfy

$$\int_\Gamma g\cdot N\,\mathrm{d}\Gamma=0.$$

Let $X=(H_0^1(\Omega))^d$ and $M=L_0^2(\Omega)=\{q/q\in L^2(\Omega), \int_\Omega q\,\mathrm{d}\Omega=0\}$; then the variational formulation of (6a)–(6d) is given as follows.

(VF) Find $u\in X$ and $p\in M$ such that

$$\int_\Omega \mathrm{grad}(u)\cdot\mathrm{grad}(v)\,\mathrm{d}\Omega-\int_\Omega p\,\mathrm{div}\,v\,\mathrm{d}\Omega=\int_\Omega f\cdot v\,\mathrm{d}\Omega\quad\forall v\in X,$$

$$\int_\Omega q\,\mathrm{div}\,u\,\mathrm{d}\Omega=0\quad\forall q\in M.$$

(See Reference 18 for the existence and uniqueness theorem for this mixed problem.)

Given the finite-dimensional subspaces $X_h\subset X$ and $M_h\subset M$, which will be specified a little later, one then obtains the following discrete variational formulation.

$(\mathrm{VF})_h$ Find $u_h\in X_h$ and $p_h\in M_h$ such that

$$\int_\Omega \mathrm{grad}(u_h)\cdot\mathrm{grad}(v_h)\,\mathrm{d}\Omega-\int_\Omega p_h\,\mathrm{div}\,v_h\,\mathrm{d}\Omega=\int_\Omega f_h\cdot v_h\,\mathrm{d}\Omega\quad\forall v_h\in X_h,$$

$$\int_\Omega q_h\,\mathrm{div}\,u_h\,\mathrm{d}\Omega=0\quad\forall q_h\in M_h.$$

This can also be written in matrix form as

$$\begin{pmatrix}A_h & B_h^{\mathrm{T}}\\ B_h & 0\end{pmatrix}\begin{Bmatrix}u_h\\ p_h\end{Bmatrix}=\begin{Bmatrix}f_h\\ 0\end{Bmatrix},\tag{7}$$

where A_h and B_h are the discrete operators respectively associated with $(-\Delta U)$ and $(-\mathrm{div})$ such that

$$a(u_h, v_h)=\int_\Omega \mathrm{grad}(u_h)\cdot\mathrm{grad}\langle v_h\rangle\,\mathrm{d}\Omega=\langle v_h\rangle^{\mathrm{T}}\,[A_h]\,\{u_h\},$$

$$b(u_h, q_h)=-\int_\Omega q_h\,\mathrm{div}\,u_h\,\mathrm{d}\Omega=\langle q_h^{\mathrm{T}}\rangle\,[B_h]\,\{u_h\}.$$

Of course, the spaces X_h and M_h cannot be chosen independently of one another in order that problem (7) has a unique solution. They must satisfy the Inf–Sup discrete condition of Babuška[5] and Brezzi.[6]

(BB) There is a constant k independent of h such that

$$\operatorname*{Inf}_{q_h\in M_h}\ \operatorname*{Sup}_{u_h\in X_h}\frac{\int_\Omega q_h B_h u_h\,\mathrm{d}\Omega}{\|u_h\|_{X_h}\|q_h\|_{M_h}}\geqslant\mu>0.$$

If the (BB) condition is not satisfied, the discretization (X_h, M_h) is unstable and the uniqueness (even existence) of a solution for problem (7) cannot be assumed (B_h is not surjective, B_h^{T} not injective). Moreover, the solution may be unique but totally unrealistic,[13,16,19,20] being still affected by the well-known spurious modes,[20] and iterative methods can be inefficient owing to the 'very bad' condition number of the operators (see References 3, 7 and 16 and Figure 3 for Q1–P0). One therefore has to filter the (so-called) spurious pressure modes which do not correspond to $p=\text{constant}$ in order to avoid these troubles, which have been clearly and extensively presented by Sani *et al.*[20] for Q1–P0.

One way of proceeding consists of adding a *regularization term* to the discrete incompressibility condition, which then becomes

$$\int_\Omega q_h \operatorname{div} u_h \, d\Omega + C_h(q_h, p_h) = 0 \quad \forall q_h \in M_h,$$

where C_h is a symmetric, bilinear and semipositive form on $M_h \times M_h$. One then obtains the following new stabilized variational formulation.

$(\mathrm{SVF})_h$ Find $u_h \in X_h$ and $p_h \in M_h$ such that

$$\int_\Omega \operatorname{grad}(u_h) \cdot \operatorname{grad}(v_h) \, d\Omega - \int_\Omega p_h \operatorname{div} v_h \, d\Omega = \int_\Omega f_h \cdot v_h \, d\Omega \quad \forall v_h \in X_h,$$

$$\int_\Omega q_h \operatorname{div} u_h \, d\Omega + C_h(q_h, p_h) = 0 \quad \forall q_h \in M_h.$$

This idea, initially proposed by Brezzi and Pitkaranta,[11] has been developed by numerous authors (see e.g. References 7, 8, 10, 12 and 14–16). Kechkar and Silvester[16] propose choosing C_h in order to satisfy the following criterion of filtering the spurious pressure modes.

(CR) $C_h(p_h, p_h) = 0$ and $\int_\Omega p_h \operatorname{div} v_h \, d\Omega = 0$ are simultaneously satisfied only by $p_h = \text{constant}$.

Then one has the existence and uniqueness of the solution (see Reference 16 for a proof).

Assume that C_h satisfies the (CR) criterion; then problem $(SVF)_h$ admits a unique solution $(u_h, p_h) \in X_h \times M_h$.

Unfortunately, no consequence can be derived from (CR) about the accuracy of the numerical results or the condition number of the operators. In Reference 16 two stabilization procedures (the 'global jump formulation' and the 'local jump formulation') have been proposed giving efficient cures to these latter troubles.

3. THE KECHKAR–SILVESTER STABILIZATION FOR THE Q1–P0 ELEMENT

Let T_h be a partitioning of Ω into quadrilaterals, one of which defines the spaces $X_h \subset X$ and $M_h \subset M$ by

$$X_h = \{V \in X \cap C^0(\Omega) / V|_K \in (Q_1(K))^d \quad \forall K \in T_h\} \quad (d = 2 \text{ or } 3),$$

$$M_h = \{p \in M / p|_K \in P_0(K) \quad \forall K \in T_h\},$$

where $Q_1(K)$ is the space of bilinear functions on K and $P_0(K)$ is the space of constant functions on K. The pair (X_h, M_h), usually referred to as the Q1–P0 finite element, does not satisfy the (BB) condition and thus provides an unstable scheme subject to spurious pressure modes p_m.

Assume that the elements in T_h have been assembled into N regular macro-elements of 2^d quadrilaterals; then p_m is the well-known *checkerboard mode*[20] ($d=2$) given by Figure 1.

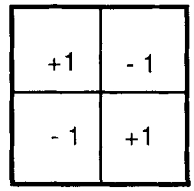

Figure 1. Spurious pressure modes on a macro-element M

Remark. There are four spurious pressure modes for $d=3$.[20]

In order to filter the spurious pressure modes, Kechkar and Silvester[16] propose the following three choices satisfying the (CR) criterion:

$$C_h(p, q)=\varepsilon \int_{\Omega} pq \, \mathrm{d}s, \tag{8a}$$

$$C_h(p, q)=\beta h \sum_{i=1}^{N_s} \int_{\partial\Gamma_i} [p]\,[q] \mathrm{d}s, \tag{8b}$$

$$C_h(p, q)=\beta h \sum_{M=1}^{N} \sum_{i=1}^{n_d} \int_{\partial\Gamma_i^M} [p]\,[q] \mathrm{d}s, \tag{8c}$$

where h denotes the mesh size (defined locally), ε is a penalization parameter ($\varepsilon>0$), β is a stabilization parameter ($\beta>0$). $[\,.\,]$ is the 'pressure jump' operator,[14,16] $\partial\Gamma_i$ ($i=1, \ldots, N_s$) are the N_S *interior* inter-element boundaries (edges in 2D, faces in 3D) and $\partial\Gamma_i^M$ ($i=1, \ldots, n_d$) are the n_d inter-element boundaries *strictly within* each macro-element M ($n_d=4$ for $d=2$ and $n_d=12$ for $d=3$).

Equations (8a)–(8c) are respectively called the consistent penalty formulation, the 'global jump formulation' and the 'local jump formulation'. The first choice (8a) is a standard penalization technique. This methodology, assuming existence and unicity of a solution, is extensively used with the Q1–P0 element,[13,19] but numerical results for the pressure field p are still affected by the checkerboard mode and no cure is obtained for the breakdown of iterative methods.[16] The second choice (8b) has been presented in References 12 and 14. The summation is over all N_S interior-element boundaries, in contrast to the choice (8c)[16] where the summation runs only over the inter-element boundaries strictly within each macro-element M.

A comparison of these techniques is done in Reference 16 showing the accuracy of the latter two choices (8b) and (8c), and in the following we choose the 'local jump formulation' (8c) because of the very attractive structure of the stabilization matrix.

$(\mathrm{SFV})_h$ can be recast in the following matrix form:

$$\begin{pmatrix} A_h & B_h^{\mathrm{T}} \\ B_h & -C_h \end{pmatrix} \begin{Bmatrix} u_h \\ p_h \end{Bmatrix} = \begin{Bmatrix} f_h \\ 0 \end{Bmatrix}. \tag{9}$$

For an appropriate numbering strategy the matrix C_h has a *block diagonal* structure where each block C_h^M corresponding to a macro-element M (see Figure 2) is of the form

$$C_h^M=\beta h^2 \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix} \quad \text{for } d=2, \tag{10}$$

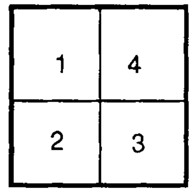

Figure 2. Numbering of quadrilaterals in a macro-element M

$$C_h^M=\beta h^3\begin{pmatrix} 3 & -1 & -1 & 0 & -1 & 0 & 0 & 0\\ -1 & 3 & 0 & -1 & 0 & -1 & 0 & 0\\ -1 & 0 & 3 & -1 & 0 & 0 & -1 & 0\\ 0 & -1 & -1 & 3 & 0 & 0 & 0 & -1\\ -1 & 0 & 0 & 0 & 3 & -1 & -1 & 0\\ 0 & -1 & 0 & 0 & -1 & 3 & 0 & -1\\ 0 & 0 & -1 & 0 & -1 & 0 & 3 & -1\\ 0 & 0 & 0 & -1 & 0 & -1 & -1 & 3 \end{pmatrix} \quad \text{for } d=3. \tag{11}$$

In the following we choose $\beta=1$, as advocated in Reference 16, and for convenience of writing, the subscript h will be omitted where this does not lead to confusion.

4. A STABILIZED CONTEXT UZAWA-TYPE ALGORITHM

4.1. Introduction

In standard Uzawa-type algorithms one deals with the dual problem (2). For stable discretization by finite elements satisfying the (BB) condition, the operator $L=BA^{-1}B^{\mathrm{T}}$ is strictly positive definite and provides a condition number bounded independently of the mesh size.[4]

Nevertheless, the matrix L is only given in an implicit way (because of the term A^{-1}) which forbids the use of all methods requiring decompositions of L, but being able to evaluate the matrix–vector product Lw for an arbitrary w is sufficient for using gradient-type methods:

$$p^{n+1}=p^n+\rho_n\,\mathbb{L}^{-1}(Lp^n-g),$$

where $\mathbb{L}$ is a preconditioner for L.

Conjugate gradient algorithms applied to problem (2) have been widely described and improved by numerous authors in the context of finite elements satisfying the (BB) Inf–Sup condition, e.g. by Verfurth[4] (multigrid solver for Poisson-type equations), Cahouet and Chabard[2] (different techniques of preconditioning) and Robichaud *et al.*[3] ('incomplete Uzawa algorithm').

In 3D good results have been presented for the enriched brick Q1–P1[3] and for the P2–P1 tetrahedron with continuous pressure,[2] but Uzawa-type algorithms have been clearly demonstrated to be inefficient for Q1–P0.[3] In the following we will show that this breakdown can be overcome by using a stabilization procedure[16] which permits us to retain the property of condition number bounded independently of h for the stabilized operator[17] $BA^{-1}B^{\mathrm{T}}+C$ and thus we obtain a less costly Uzawa algorithm owing to the low storage and computational requirements of Q1–P0 (especially in the 3D case).

In order to enhance the convergence factor of the conjugate gradient method, we describe in Section 4.3 a new preconditioner[17] making the greatest use of the block diagonal structure of the stabilization matrix C.

Remark. At every iteration step the evaluation of $Lp^n = BA^{-1}B^Tp^n$ leads to the computation of $z^n = A^{-1}B^Tp^n$, which must be interpreted as the solution of a Poisson-type problem

$$Az^n = B^Tp^n. \tag{12}$$

An easy way is to use Choleski decomposition of the matrix A, which can be rather less expensive owing to the formulation of $a(.)$ and the choice of a low-order discretization. In effect, A is obtained by reproducing for each component of the velocity the discrete scalar Laplacian corresponding to the bilinear element. Naturally, for a symmetric stress tensor or augmented Lagrangian formulation the components are then coupled, so it can be of major interest to use iterative solvers.[3,4]

4.2. Breakdown of the non-stabilized algorithm

For the non-stabilized dual problem (2),

$$Lp = (BA^{-1}B^T)p = BA^{-1}f = g,$$

we obtain the standard conjugate gradient Uzawa algorithm.[1] Unfortunately, for the Q1–P0 finite element discretization, L is only semipositive (Ker (L) contains the spurious pressure mode given in Figure 1). However, if (2) admits a solution, we can obtain a convergent conjugate gradient method. In effect, we recall the following result.[21]

> Assume that $M \geqslant 0$ and $c \in \mathrm{Im}(M)$; then $Mx = c$ admits a unique solution $x^* \in \mathrm{Im}\,(M)$ and the conjugate gradient iterated term x^n converges to x^*.

For a 'well-posed' problem (in the sense that $0 \in \mathrm{Im}(B)$) the proposed conjugate gradient method leads to the curves of the evolution of the residual $\| Lp^n - g \|$ presented in Figure 3 for different values of the mesh size h and for 2D and 3D computations.

It clearly appears that the convergence factor of this method degrades significantly when the problem size decreases (see also Reference 3), owing to the instability of the Q1–P0 finite element discretization leading to an ill-conditioned matrix $L(O(h^{-2}))$.[19,22] Thus the necessity for a stabilization procedure is evident.

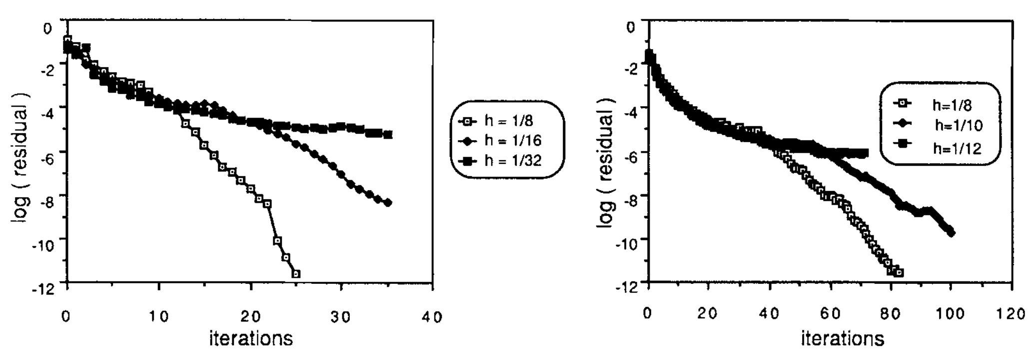

Figure 3. Evolution of the residual $\| Lp^n - g \|$ in the non-stabilized case: (a) 2D; (b) 3D

4.3. Efficiency of the stabilized algorithm

4.3.1. Properties of the stabilized dual operator. Eliminating the velocity vector u, problem (9) may be reformulated as the corresponding *stabilized dual problem* for the pressure:

$$L_S p = (BA^{-1}B^T + C)p = BA^{-1}f = g. \tag{13}$$

As an immediate consequence of the (CR) criterion satisfied by C, the symmetric matrix L_S is positive definite on M_h:

$$\forall q \in M_h \ (q \neq 0) \quad q^T L_S q = q^T(BA^{-1}B^T)q + q^T C q$$
$$= (B^T q)^T A^{-1}(B^T q) + q^T C q > 0,$$

since $B^T q$ and $q^T C q$ are simultaneously zero only for q = constant from (CR). Thus a conjugate gradient method is applied to (13) and we retain the property that the convergence factor is bounded independently of the mesh size, as clearly seen in Figure 4.

We think, supported by the comparison Q1–P0/Q1$^+$–P1 of Robichaud *et al.*,[3] that this result is very important. With a simple stabilization procedure one can efficiently cure the bad properties of an unstable discretization and thus provide an efficient Uzawa algorithm, up to now missing, for the less costly and very popular Q1–P0.

4.3.2. Construction of a preconditioner. It is important to recall that the operator $L = BA^{-1}B^T$ (or $L_S = BA^{-1}B^T + C$) is only given in an implicit way (because of A^{-1}) and thus forbids all decompositions of L as preconditioner.

Cahouet and Chabard[2] propose different preconditioners for the dual formulation of the steady or unsteady quasi-Stokes problem. A direct application of their techniques in our steady stabilized case leads to the following preconditioner:

$$\mathbb{L}_S = B\bar{A}^{-1}B^T + C, \quad \text{where } \bar{A} = \text{diag}(A). \tag{14}$$

In order to take into account the block diagonal structure of the matrix C(which is not the case with $B\bar{A}^{-1}B^T$), we propose the new preconditioner[17]

$$\bar{L}_S = \text{diag}(B\bar{A}^{-1}B^T) + C, \quad \text{where } \bar{A} = \text{diag}(A). \tag{15}$$

Such a preconditioner, taking full advantage of the structure of $\bar{L}_S$, is much less costly, since it reduces to the solution of N independent subsystems of small size: 4×4 for $d = 2$ and 8×8 for

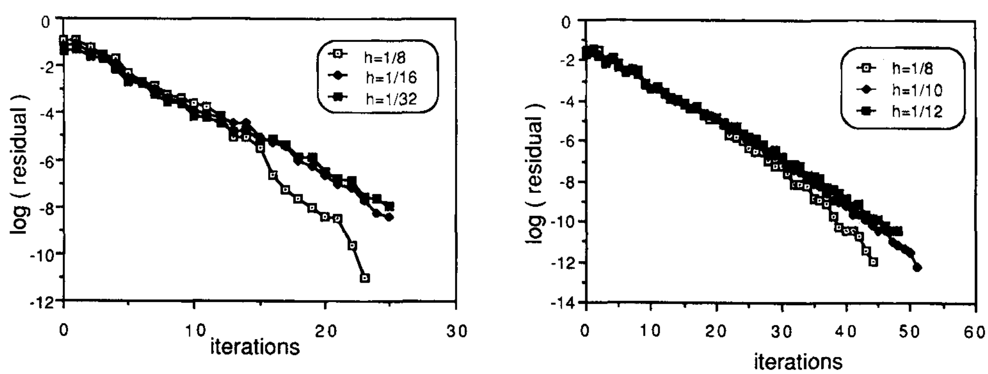

Figure 4. Evolution of the residual $\| L_S p^n - g \|$ in the stabilized case: (a) 2D; (b) 3D

$d=3$ (N is the number of macro-elements). Moreover, the independence of these systems can be used on parallel computers.

This choice of preconditioner is supported by the performances presented in Figure 5 and its efficiency seems to be improved for 3D computations.

Remark. For convenience of writing in the legends of the figures we use (NS) for 'non-stabilized', (SNP) for 'stabilized but not preconditioned' and (SP) for 'stabilized and preconditioned'.

A measure of improvement is proposed in Figure 6, where the reader can appreciate the benefits coming from the stabilization contribution and from the preconditioning one.

5. CONCLUSIONS

A stabilization procedure has permitted efficient use of a Uzawa-type method for a 'low-order' discretization of the Stokes problem with finite elements not satisfying the Babuška–Brezzi Inf–Sup condition. With benefits from the stabilized context we have retained the usual result (for stable discretization) that the convergence factor is bounded independently of the mesh size, and a preconditioner profiting from the block diagonal structure of the stabilization matrix has led to good results. This approach, up to now missing, has permitted us to use one of the 'cheapest' and most popular elements for 3D problems, i.e. the Q1–P0 element. These techniques, initially used in 2D computations,[17] have been extended to 3D problems (see Reference 16 for the 3D

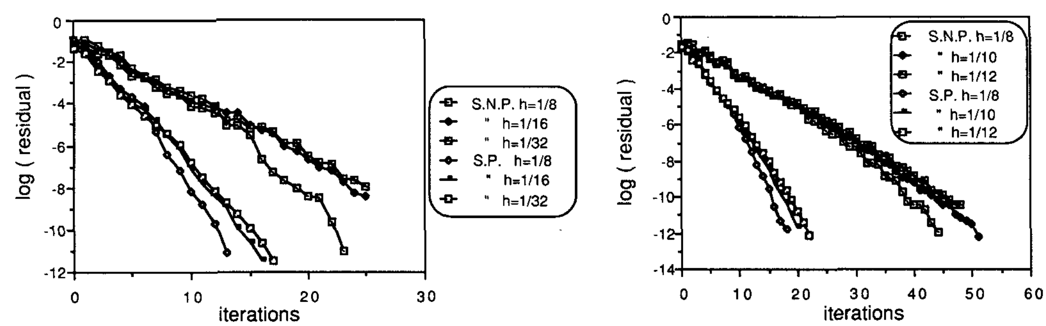

Figure 5. Efficiency of the preconditioner in the stabilized case: (a) 2D; (b) 3D

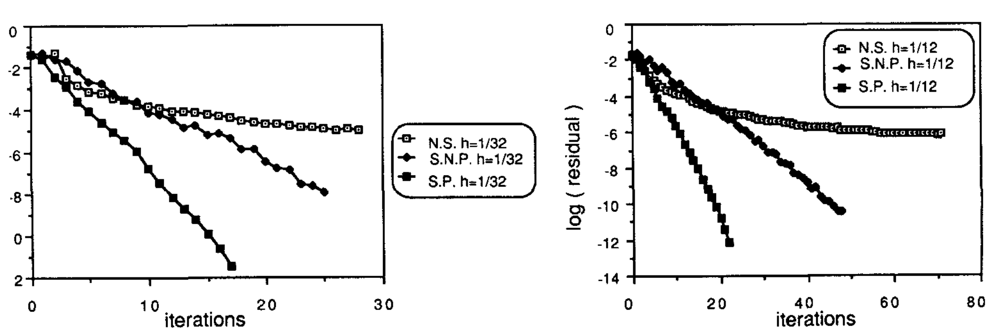

Figure 6. Measure of improvement: (a) 2D; (b) 3D

stabilization procedure) and we hope now to utilize them efficiently in the setting of a class of more general methods, the so-called 'incomplete Uzawa algorithm.[3] The first results obtained by a 'complete Uzawa-type' method have been very encouraging.

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